Phys 410
Spring 2013
Lecture #20 Summary
8 March, 2013

We began with the Brachistochrone problem. A particle falls from rest under the influence of gravity following a frictionless track to a final location. The question is: what track design will get the particle to the final location in the shortest time? The particle starts at the origin (x=0, y=0) and falls to a point (x_2, y_2), with $x_2 > 0$ and $y_2 > 0$. The time to travel is given by $Time(1 \rightarrow 2) = \int_1^2 dt = \int_1^2 \frac{ds}{v} = \int_1^2 \frac{\sqrt{dx^2+dy^2}}{v}$. The speed is found from conservation of energy: $v = \sqrt{2gy}$, leading to $Time(1 \rightarrow 2) = \frac{1}{\sqrt{2g}} \int_0^{y_2} \frac{\sqrt{1+(x\prime)^2}}{\sqrt{y}} dy$, where we are using the y-coordinate of the particle as the independent variable and $x' = dx/dy$. This integral will be made stationary when the integrand $f(x, x', y)$ obeys the Euler-Lagrange equation, which in this case is: $\frac{\partial f}{\partial x} - \frac{d}{dy}\frac{\partial f}{\partial x'} = 0$. The result is a differential equation: $x' = \sqrt{\frac{y}{2a-y}}$, where a is a constant introduced from the Euler-Lagrange equation. We can integrate this equation with the change of variables $y = a(1 - \cos\theta)$, yielding $x = a(\theta - \sin\theta) + C$. This describes a cycloid curve. The particle making the shortest fall will follow the cycloid trajectory.

In general, it is not always possible to parameterize the trajectory of a particle with a simple one-to-one functional relationship such as $y(x)$ or $x(y)$. In this case one would like to parameterize the trajectory with functions such as $x(u), y(u)$, where u acts as the parameter. The Euler-Lagrange equation can be generalized to handle this situation. Consider the integral $S = \int_{u_1}^{u_2} f[x(u), x'(u), y(u), y'(u), u]du$. To make it stationary will yield two equations: $\frac{\partial f}{\partial x} - \frac{d}{du}\frac{\partial f}{\partial x'} = 0$ and $\frac{\partial f}{\partial y} - \frac{d}{du}\frac{\partial f}{\partial y\prime} = 0$.

We then showed that Newton's second law of motion can be re-stated as a set of Euler-Lagrange equations for an integrand known as the Lagrangian $\mathcal{L} = T - U$, where T is the kinetic energy and U is the potential energy. The integral that is made stationary is called the action: $S = \int \mathcal{L}\, dt$. The Lagrangian can be written in terms of any set of unique (generalized) coordinates (q_1, q_2, q_3). One can define a generalized force as $\frac{\partial \mathcal{L}}{\partial q_i}$, and the generalized momentum as $\frac{\partial \mathcal{L}}{\partial q_i'}$. They are related as "generalized force" = time rate of change of "generalized momentum". These generalized quantities do not necessarily have the dimensions of force or momentum.

Finally we considered the Lagrangian in polar coordinates for a particle acted upon by a conservative force in two dimensions. The Lagrangian is $\mathcal{L}(r, \dot{r}, \phi, \dot{\phi}, t) = \frac{m}{2}(\dot{r}^2 + r^2\dot{\phi}^2) - U(r, \phi)$. The Euler-Lagrange equation for r yields $-\frac{\partial U}{\partial r} = m(\ddot{r} - r\dot{\phi}^2)$. This is Newton's second law for radial motion, where the first term on the right hand side is the radial acceleration, while the second is the centripetal acceleration. The Euler-Lagrange equation for φ yields $-\frac{\partial U}{\partial \varphi} = \frac{d}{dt}(mr^2\dot{\varphi})$. This is a statement that the torque acting on the particle $\left(-\frac{\partial U}{\partial \varphi} = rF_\varphi\right)$ is equal to the time rate of change of the angular momentum. In other words it is a statement of Newton's second law for rotational motion.